

Aging properties of an anomalously diffusing particule

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Abstract

We report new results about the two-time dynamics of an anomalously diffusing classical particle, as described by the generalized Langevin equation with a frequency-dependent noise and the associated friction. The noise is defined by its spectral density proportional to $\omega^{\delta-1}$ at low frequencies, with $0 < \delta < 1$ (subdiffusion) or $1 < \delta < 2$ (superdiffusion). Using Laplace analysis, we derive analytic expressions in terms of Mittag-Leffler functions for the correlation functions of the velocity and of the displacement. While the velocity thermalizes at large times (slowly, in contrast to the standard Brownian motion case $\delta = 1$), the displacement never attains equilibrium: it ages. We thus show that this feature of normal diffusion is shared by a subdiffusive or superdiffusive motion. We provide a closed form analytic expression for the fluctuation-dissipation ratio characterizing aging.

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1. Introduction

In this paper, we study the two-time dynamics and the aging properties of a classical particle diffusing in the presence of coloured (*i.e.* frequency-dependent) noise and friction. The dynamics of the particle is governed by the generalized Langevin equation [1]–[2]. We will not be concerned here with the specific microscopic origin of the noise and of the friction. The knowledge of the noise spectral density suffices to make the model precise. The colored noise is conveniently defined by its spectral density, assumed to be proportional to $\omega^{\delta-1}$ at low frequencies with $0 < \delta < 1$ or $1 < \delta < 2$. The frequency-dependent friction is deduced from the noise by means of the second fluctuation-dissipation theorem (FDT).

This general framework can be used in the classical domain as well as in the quantum one, depending on which formulation is chosen for the FDT. The one-time dynamics of this dissipative model has been extensively studied, with particular emphasis on the quantum case [3]–[6]. In the classical case, the model allows for the description of anomalous diffusion, that is, either subdiffusion (for $0 < \delta < 1$) or superdiffusion (for $1 < \delta < 2$). Closed analytic expressions of the lowest moments of the velocity and of the displacement of the anomalously diffusing particle have been obtained in [7].

The two-time dynamics of the diffusing particle has been investigated in the case $\delta = 1$, which corresponds to white noise, non-retarded friction and standard Brownian motion [8]–[9]. When studying the two-time correlation functions of the dynamical variables attached to the particle, one is faced with interesting new physical effects such as aging, *i.e.* the absence of time-translation invariance. Indeed, as first noticed in [8], the two-time correlation function $\langle x(t)x(t') \rangle$ of the displacement of the diffusing particle depends explicitly on both times t and t' and not only on the difference $t - t'$, even in the limit of large age t' ($0 \leq t' \leq t$). In contrast to the velocity, which equilibrates at large times, and does not age, the displacement is an out-of-equilibrium variable. The equilibrium FDT has to be modified in order to relate the displacement response and correlation functions. This can be achieved through the introduction of a factor (the fluctuation-dissipation ratio) rescaling the temperature [8]. We have demonstrated in [9] that, for a classical diffusing particle, the fluctuation-dissipation ratio can be expressed in terms of the time-dependent diffusion coefficients $D(\tau)$ and $D(t_w)$, where $\tau = t - t'$ denotes the

observation time and $t_w = t'$ the waiting time. We have subsequently extended this result to non integer values of δ between 0 and 2 [10].

In the present paper, we study the two-time dynamics of an anomalously diffusing particle. Anomalous diffusion, which has been observed in various physical systems, is the subject of a growing interest, both experimental and theoretical (for a recent review see [11]). Making extensive use of Laplace analysis, we derive closed form analytic expressions for the two-time correlation functions of the velocity and of the displacement. We demonstrate that these two-time quantities are expressible in terms of Mittag-Leffler functions [12], which have previously been shown to play a central role in the relaxational one-time dynamics of the particle [7]. As a result, while the particle velocity thermalizes at large times (slowly, except for $\delta = 1$), its displacement never attains equilibrium: it ages. We thus show that this feature of standard Brownian motion is shared by a subdiffusive or superdiffusive particle. We provide closed form analytic expressions in terms of Mittag-Leffler functions for the time-dependent diffusion coefficients $D(\tau)$ and $D(t_w)$, as well as for the fluctuation-dissipation ratio characterizing aging.

2. Generalized Langevin description of anomalous diffusion: a Laplace analysis

2.1. The model and the dynamical variables of interest

The generalized Langevin equation for a classical particle of mass m diffusing in the absence of a deterministic potential writes

$$m \frac{dv}{dt} + \int_0^t \gamma(t-t')v(t') dt' = F(t), \quad v = \frac{dx}{dt}. \quad (2.1)$$

In Eq. (2.1), $F(t)$ is the Langevin force acting on the particle, as modeled by a stationary Gaussian random process of zero mean, and $\gamma(t)$ is a retarded friction kernel, which is an even function of t [1]–[2]. As well-known, the coherence of the model implies that $F(t)$ and $\gamma(t)$ are not independent quantities, but instead are linked by the second fluctuation-dissipation theorem, that is

$$\langle F(t)F(t') \rangle = kTm\gamma(|t-t'|), \quad (2.2)$$

where T is the temperature. (The symbol $\langle \dots \rangle$ denotes the average over the realizations of the noise). For white noise, one simply has

$$\langle F(t)F(t') \rangle = 2kT\eta \delta(t-t'), \quad \gamma(t) = 2\gamma \delta(t), \quad (2.3)$$

where γ is the friction coefficient and $\eta = m\gamma$ the viscosity. The Langevin equation is then non-retarded:

$$m \frac{dv}{dt} + m\gamma v = F(t). \quad (2.4)$$

In the following, we shall be interested with the one-time and two-time properties of two dynamical variables attached to the particle, namely its velocity as defined by the solution of Eq. (2.1) satisfying the initial condition

$$v(t = 0) = v_0, \quad (2.5)$$

and its displacement, as defined by

$$x(t) = \int_0^t v(t') dt'. \quad (2.6)$$

We will primarily be concerned with non-white noise of parameters to be specified below and the associated friction coefficient. Let us beforehand describe the general lines of our analysis, which heavily relies upon the use of Laplace transformation.

2.2. Laplace analysis

The one-time and the two-time properties of the dynamical variables of interest can conveniently be studied by means of Laplace analysis, which implies that all the quantities of interest are to be considered as causal functions. Defining, for any quantity $q(t)$, its Laplace transform $\hat{q}(z) = \int_0^\infty q(t) e^{-zt} dt$, one gets, by applying the Laplace transformation to the generalized Langevin equation (2.1):

$$mz\hat{v}(z) + m\hat{\gamma}(z)\hat{v}(z) = \hat{F}(z) + mv(t = 0). \quad (2.7)$$

2.3. Statistical properties of $\hat{F}(z)$

The generalized Langevin model is fully specified by Eq. (2.7) with the initial condition (2.5), together with the given statistical properties of $\hat{F}(z)$. Since $F(t)$ is a Gaussian process, the statistical properties of $\hat{F}(z)$ are completely characterized by the average value $\langle \hat{F}(z) \rangle$ and the correlation function $\langle \hat{F}(z)\hat{F}(z') \rangle$.

The Langevin force $F(t)$ being a process of zero mean, one has:

$$\langle \hat{F}(z) \rangle = 0. \quad (2.8)$$

As for the correlation function $\langle \hat{F}(z)\hat{F}(z') \rangle$, one gets from the fluctuation-dissipation relation (2.2):

$$\langle \hat{F}(z)\hat{F}(z') \rangle = kTm \int_0^\infty \int_0^\infty e^{-z't'} e^{-zt} \gamma(|t-t'|) dt dt'. \quad (2.9)$$

To compute the double integral (2.9), we shall consider separately the two contributions $\langle \hat{F}(z)\hat{F}(z') \rangle_1$ and $\langle \hat{F}(z)\hat{F}(z') \rangle_2$, coming respectively from the time domains $t' \leq t$ and $t \leq t'$. Let us first consider the contribution from the domain $t' \leq t$. Setting $u = t - t'$, one gets

$$\langle \hat{F}(z)\hat{F}(z') \rangle_1 = kTm \int_0^\infty e^{-(z+z')t'} dt' \int_0^\infty e^{-zu} \gamma(u) du, \quad (2.10)$$

that is:

$$\langle \hat{F}(z)\hat{F}(z') \rangle_1 = kTm \frac{1}{z+z'} \hat{\gamma}(z). \quad (2.11)$$

Coming then to the contribution from the domain $t \leq t'$, one gets

$$\langle \hat{F}(z)\hat{F}(z') \rangle_2 = kTm \int_0^\infty e^{-(z+z')t} dt \int_0^\infty e^{-z'v} \gamma(v) dv, \quad (2.12)$$

that is:

$$\langle \hat{F}(z)\hat{F}(z') \rangle_2 = kTm \frac{1}{z+z'} \hat{\gamma}(z'). \quad (2.13)$$

Adding the contributions (2.11) and (2.13), one finally obtains:

$$\langle \hat{F}(z)\hat{F}(z') \rangle = kTm \frac{\hat{\gamma}(z) + \hat{\gamma}(z')}{z+z'}. \quad (2.14)$$

Eq. (2.14), which represents the Laplace domain formulation of the second FDT (Eq. (2.2)), will play a central role in the following.

Actually, relations of the form (2.14) hold quite generally. Indeed, given any stationary correlation function of the form

$$\langle \phi(t)\phi(t') \rangle = A f(|t-t'|), \quad (2.15)$$

one can show in a similar way that its double Laplace transform writes

$$\langle \hat{\phi}(z)\hat{\phi}(z') \rangle = A \frac{\hat{f}(z) + \hat{f}(z')}{z+z'}. \quad (2.16)$$

2.4. Statistical properties of $\hat{v}(z)$

One deduces from Eqs. (2.7) and (2.5) the expression of $\hat{v}(z)$:

$$\hat{v}(z) = \frac{\hat{F}(z)}{z + \hat{\gamma}(z)} + \frac{v_0}{z + \hat{\gamma}(z)}. \quad (2.17)$$

As for the average particle velocity, one has, since $\langle \hat{F}(z) \rangle = 0$:

$$\langle \hat{v}(z) \rangle = \frac{v_0}{z + \hat{\gamma}(z)}. \quad (2.18)$$

The double Laplace transform $\langle \hat{v}(z)\hat{v}(z') \rangle$ of the two-time velocity correlation function $\langle v(t)v(t') \rangle$ is given by:

$$\langle \hat{v}(z)\hat{v}(z') \rangle = \frac{\langle \hat{F}(z)\hat{F}(z') \rangle}{m^2[z + \hat{\gamma}(z)][z' + \hat{\gamma}(z')]} + \frac{v_0^2}{[z + \hat{\gamma}(z)][z' + \hat{\gamma}(z')]} \quad (2.19)$$

It is the sum of two contributions, the first one coming from the random force correlation function, the second one being linked to the initial condition. Taking into account the expression (2.14) of $\langle \hat{F}(z)\hat{F}(z') \rangle$, one can rewrite Eq. (2.19) as

$$\langle \hat{v}(z)\hat{v}(z') \rangle = \frac{kT}{m} \frac{\hat{K}(z) + \hat{K}(z')}{z + z'} + \left(v_0^2 - \frac{kT}{m} \right) \hat{K}(z) \hat{K}(z'), \quad (2.20)$$

with $\hat{K}(z)$ as defined by

$$\hat{K}(z) = \frac{1}{z + \hat{\gamma}(z)}. \quad (2.21)$$

Formula (2.20) displays clearly the fact that the correlation function $\langle \hat{v}(z)\hat{v}(z') \rangle$ is the sum of two contributions of a different character as far as stationarity properties are concerned. The first one, $\frac{kT}{m} \frac{\hat{K}(z) + \hat{K}(z')}{z + z'}$, of the general form (2.16), corresponds in the time domain to a stationary random process. The second one, $(v_0^2 - \frac{kT}{m}) \hat{K}(z) \hat{K}(z')$, corresponds to a function depending separately on t and t' (and not only on the difference $t - t'$). This term, if non-zero, will yield an aging contribution to the two-time velocity correlation function $\langle v(t)v(t') \rangle$.

Averaging out Eq. (2.20) over an equilibrium ensemble of initial velocities, the aging contribution to $\langle v(t)v(t') \rangle$ disappears. We shall come back to this point later. One is then left with:

$$\langle \hat{v}(z)\hat{v}(z') \rangle = \frac{kT}{m} \frac{\hat{K}(z) + \hat{K}(z')}{z + z'}. \quad (2.22)$$

2.5. Statistical properties of $\hat{x}(z)$

One deduces from Eq. (2.6) the expression of $\hat{x}(z)$:

$$\hat{x}(z) = \frac{1}{z} \hat{v}(z). \quad (2.23)$$

As for the average particle displacement, one has:

$$\langle \hat{x}(z) \rangle = \frac{1}{z} \frac{v_0}{z + \hat{\gamma}(z)}. \quad (2.24)$$

We assume that the particle velocity is at thermal equilibrium and does not age. The double Laplace transform $\langle \hat{x}(z)\hat{x}(z') \rangle$ of the two-time displacement correlation function $\langle x(t)x(t') \rangle$ is given by:

$$\langle \hat{x}(z)\hat{x}(z') \rangle = \frac{kT}{m} \frac{1}{z} \frac{1}{z'} \frac{\hat{K}(z) + \hat{K}(z')}{z + z'}. \quad (2.25)$$

Due to the presence of the factors $1/z$ and $1/z'$ in the r.h.s. of Eq. (2.25), the correlation function $\langle \hat{x}(z)\hat{x}(z') \rangle$ is not of the form (2.16). Consequently, in the time domain, the two-time displacement correlation function $\langle x(t)x(t') \rangle$ will depend separately on t and t' . The particle displacement is thus an aging variable.

Let us now be more specific and introduce the parameters of the so-called non-Ohmic noise and friction allowing for the description of the dynamics of a subdiffusive or superdiffusive particle.

3. Noise and friction of the non-Ohmic Langevin model

We consider a coloured noise of spectral density

$$\langle |F(\omega)|^2 \rangle = 2kT\eta_\delta \left(\frac{|\omega|}{\tilde{\omega}} \right)^{\delta-1} f_c \left(\frac{|\omega|}{\omega_c} \right). \quad (3.1)$$

The small- $|\omega|$ behaviour of $\langle |F(\omega)|^2 \rangle$ is a power-law characterized by the exponent $\delta - 1$. The function $f_c(|\omega|/\omega_c)$ is a high-frequency cut-off function of typical width ω_c , and $\tilde{\omega} \ll \omega_c$ denotes a reference frequency allowing for the coupling constant $\eta_\delta = m\gamma_\delta$ to have the dimension of a viscosity for any δ [6]. For $\delta = 1$, the noise spectral density is a constant (white-noise), at least in the frequency range $|\omega| \ll \omega_c$. Then the Langevin force is delta-correlated and the Langevin equation is non-retarded. The white-noise case corresponds to Ohmic friction. The cases $0 < \delta < 1$ and $\delta > 1$ are known respectively as the sub-Ohmic and super-Ohmic

models. Here we will assume that $0 < \delta < 2$, for reasons to be developed below [3]–[6].

To begin with, we have to compute the correlation function $\langle \hat{F}(z)\hat{F}(z') \rangle$. We first deduce from Eq. (3.1) the expression of the two-time correlation function $\langle F(t)F(t') \rangle$ by using the Wiener-Khintchine theorem:

$$\langle F(t)F(t') \rangle = 2kT\eta \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{|\omega|}{\tilde{\omega}} \right)^{\delta-1} f_c\left(\frac{|\omega|}{\omega_c}\right) \cos \omega(t-t'). \quad (3.2)$$

Then, one deduces from Eq. (3.2) the expression of the double Laplace transform $\langle \hat{F}(z)\hat{F}(z') \rangle$:

$$\langle \hat{F}(z)\hat{F}(z') \rangle = 2kT\eta_\delta \frac{1}{z+z'} \int_0^\infty \frac{d\omega}{\pi} \left(\frac{\omega}{\tilde{\omega}} \right)^{\delta-1} f_c\left(\frac{\omega}{\omega_c}\right) \left(\frac{z}{z^2+\omega^2} + \frac{z'}{z'^2+\omega^2} \right). \quad (3.3)$$

This expression is actually of the form (2.14), with

$$\hat{\gamma}(z) = \gamma_\delta \int_0^\infty \frac{d\omega}{\pi} \left(\frac{\omega}{\tilde{\omega}} \right)^{\delta-1} f_c\left(\frac{\omega}{\omega_c}\right) \frac{z}{z^2+\omega^2}. \quad (3.4)$$

Since $0 < \delta < 2$, one can safely make the limit $\omega_c \rightarrow \infty$ in formulas (3.3) and (3.4), and write

$$\hat{\gamma}(z) = \gamma_\delta \int_0^\infty \frac{d\omega}{\pi} \left(\frac{\omega}{\tilde{\omega}} \right)^{\delta-1} \frac{z}{z^2+\omega^2}, \quad (3.5)$$

and

$$\langle \hat{F}(z)\hat{F}(z') \rangle = 2kT\eta_\delta \frac{1}{z+z'} \int_0^\infty \frac{d\omega}{\pi} \left(\frac{\omega}{\tilde{\omega}} \right)^{\delta-1} \left(\frac{z}{z^2+\omega^2} + \frac{z'}{z'^2+\omega^2} \right). \quad (3.6)$$

The integrations over ω in formulas (3.5) and (3.6) once carried out, one gets

$$\hat{\gamma}(z) = \gamma_\delta \left(\frac{z}{\tilde{\omega}} \right)^{\delta-1} \frac{1}{\sin \frac{\delta\pi}{2}}, \quad (3.7)$$

and

$$\langle \hat{F}(z)\hat{F}(z') \rangle = kT\eta_\delta \frac{1}{z+z'} \left\{ \left(\frac{z}{\tilde{\omega}} \right)^{\delta-1} + \left(\frac{z'}{\tilde{\omega}} \right)^{\delta-1} \right\} \frac{1}{\sin \frac{\delta\pi}{2}}. \quad (3.8)$$

At this stage, following [6], it is convenient to introduce the δ -dependent frequency ω_δ as defined by

$$\omega_\delta^{2-\delta} = \gamma_\delta \frac{1}{\tilde{\omega}^{\delta-1}} \frac{1}{\sin \frac{\delta\pi}{2}}. \quad (3.9)$$

Using this notation, one has

$$\hat{\gamma}(z) = \omega_\delta^{2-\delta} z^{\delta-1}, \quad (3.10)$$

and

$$\langle \hat{F}(z) \hat{F}(z') \rangle = kTm \omega_\delta^{2-\delta} \left(\frac{z^{\delta-1} + z'^{\delta-1}}{z + z'} \right). \quad (3.11)$$

Eqs. (3.10) and (3.11) together with the Langevin equation (2.7) and the initial condition (2.5) fully specify the model in the Laplace domain.

4. Particle velocity: one-time and two-time properties

4.1. The average velocity

By inverting the Laplace transformation, one gets from Eq. (2.18):

$$\langle v(t) \rangle = v_0 \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{1}{z + \hat{\gamma}(z)} e^{zt} dz. \quad (4.1)$$

The constant c in the definition of the integration contour in Eq. (4.1) is real and chosen such that all the singularities of $\hat{K}(z) = 1/[z + \hat{\gamma}(z)]$ are lying to the left of the integration path. Using the expression (3.10) of $\hat{\gamma}(z)$, one is left with

$$\langle v(t) \rangle = v_0 \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{1}{z + \omega_\delta^{2-\delta} z^{\delta-1}} e^{zt} dz, \quad (4.2)$$

i.e., as showed in [6]–[7]:

$$\langle v(t) \rangle = v_0 E_{2-\delta}[-(\omega_\delta t)^{2-\delta}]. \quad (4.3)$$

Eq. (4.3) displays the fact that the average velocity relaxes towards zero, this decay being described by the Mittag-Leffler function¹ $E_\alpha(x)$ [12] with $\alpha = 2 - \delta$ and $x = -(\omega_\delta t)^{2-\delta}$ [6]–[7].

¹ The Mittag-Leffler function is defined by the series expansion

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad (4.4)$$

where Γ is the Euler Gamma function. The Mittag-Leffler function $E_\alpha(x)$ reduces to the exponential e^x when $\alpha = 1$. The asymptotic behaviour at large x of the Mittag-Leffler function $E_\alpha(x)$ is as follows:

$$E_\alpha(x) \simeq -\frac{1}{x} \frac{1}{\Gamma(1-\alpha)}, \quad x \gg 1. \quad (4.5)$$

At large times (i.e. $\omega_\delta t \gg 1$), the average particle velocity decreases according to a power-law of time:

$$\langle v(t) \rangle \simeq v_0 \frac{(\omega_\delta t)^{\delta-2}}{\Gamma(\delta-1)}, \quad 0 < \delta < 2, \quad \delta \neq 1. \quad (4.6)$$

In the Ohmic case $\delta = 1$, one has $\omega_{\delta=1} = \gamma$. Eq. (4.3) is then nothing but the standard exponential decay of the Brownian particle average velocity:

$$\langle v(t) \rangle = v_0 e^{-\gamma t}. \quad (4.7)$$

4.2. The velocity correlation function

Eq. (2.20) yields in the time domain

$$\begin{aligned} \langle v(t)v(t') \rangle &= \frac{kT}{m} E_{2-\delta} [-(\omega_\delta |t - t'|^{2-\delta})] \\ &\quad + \left(v_0^2 - \frac{kT}{m} \right) E_{2-\delta} [-(\omega_\delta t)^{2-\delta}] \times E_{2-\delta} [-(\omega_\delta t')^{2-\delta}]. \end{aligned} \quad (4.8)$$

As expected, the two-time velocity correlation function $\langle v(t)v(t') \rangle$ is the sum of a stationary part and of an aging one. In the Ohmic case $\delta = 1$, Eq. (4.8) reduces to:

$$\langle v(t)v(t') \rangle = \frac{kT}{m} e^{-\gamma|t-t'|} + \left(v_0^2 - \frac{kT}{m} \right) e^{-\gamma(t+t')}. \quad (4.9)$$

As already noted, the aging part of $\langle v(t)v(t') \rangle$ vanishes if one carries out an average of Eq. (4.8) over an equilibrium ensemble of initial velocities. Equivalently, this can also be achieved if the initial time, instead of being taken equal to 0, is taken equal to t_i , and subsequently rejected to $-\infty$. The aging part of $\langle v(t)v(t') \rangle$, which is then equal to the product $E_{2-\delta}(-[\omega_\delta(t-t_i)]^{2-\delta}) \times E_{2-\delta}(-[\omega_\delta(t'-t_i)]^{2-\delta})$, indeed vanishes in the limit $t_i \rightarrow -\infty$. In this limit, any initial fluctuation of the average velocity has decayed to zero at finite time. The particle velocity is then at equilibrium. Correspondingly, the two-time velocity correlation function reduces to its stationary part:

$$\langle v(t)v(t') \rangle = \frac{kT}{m} E_{2-\delta} [-(\omega_\delta |t - t'|)^{2-\delta}]. \quad (4.10)$$

Interestingly enough, the stationary part of $\langle v(t)v(t') \rangle$ is proportional to the Mittag-Leffler function $E_{2-\delta} [-(\omega_\delta |t - t'|)^{2-\delta}]$ (Eq. (4.10)), while the average

velocity $\langle v(t) \rangle$ resulting from a given initial fluctuation evolves proportionally to $E_{2-\delta}[-(\omega_\delta t)^{2-\delta}]$ (Eq. (4.3)). This fact constitutes the generalization to the non-Ohmic case ($\delta \neq 1$) of the regression theorem according to which the fluctuations decay in time following the same law as the average value, as it is well-known for the exponential decay of fluctuations which holds in standard Brownian motion ($\delta = 1$).

In the particular case $t = t'$, one gets from Eq. (4.8)

$$\langle v^2(t) \rangle = \frac{kT}{m} + \left(v_0^2 - \frac{kT}{m} \right) \left\{ E_{2-\delta}[-(\omega_\delta t)^{2-\delta}] \right\}^2, \quad (4.11)$$

a result which describes the slow relaxation of the second moment of the velocity towards its equilibrium value [7].

5. Aging of the particle displacement

From now on, we assume that the two-time velocity correlation function reduces to its stationary part (4.10). The particle velocity is thermalized. Then, the two-time displacement correlation function can directly be obtained by Laplace inversion of the correlation function $\langle \hat{x}(z)\hat{x}(z') \rangle$ as given by Eq. (2.25). One thus gets, for $0 \leq t' \leq t$:

$$\begin{aligned} \langle x(t)x(t') \rangle = \frac{kT}{m} \bigg\{ t^2 E_{2-\delta,3}[-(\omega_\delta t)^{2-\delta}] + t'^2 E_{2-\delta,3}[-(\omega_\delta t')^{2-\delta}] \\ - (t-t')^2 E_{2-\delta,3}(-[\omega_\delta(t-t')]^{2-\delta}) \bigg\}, \end{aligned} \quad (5.1)$$

where we have used the generalized Mittag-Leffler function² $E_{\alpha,\beta}(x)$ [12]. Another way of deriving the result (5.1) is to compute the double integral

$$\langle x(t)x(t') \rangle = \int_0^t \int_0^{t'} \langle v(t_1)v(t_2) \rangle dt_1 dt_2 \quad (5.3)$$

the correlation function $\langle v(t_1)v(t_2) \rangle$ being taken from Eq. (4.10). One has to use the integration formulas

$$\int_0^t E_{2-\delta}[-(\omega_\delta t_1)^{2-\delta}] dt_1 = t E_{2-\delta,2}[-(\omega_\delta t)^{2-\delta}] \quad (5.4)$$

² The generalized Mittag-Leffler function is defined by the series expansion

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \quad \beta > 0. \quad (5.2)$$

and

$$\int_0^t dt_1 \int_0^{t_1} E_{2-\delta} [-(\omega_\delta t_2)^{2-\delta}] dt_2 = t^2 E_{2-\delta,3} [-(\omega_\delta t)^{2-\delta}]. \quad (5.5)$$

Eq. (5.1) displays the fact that the particle displacement is not a variable at equilibrium. When $\delta = 1$, we recover the result we previously obtained in [9]:

$$\langle x(t)x(t') \rangle = \frac{kT}{\eta} \left(2t' - \frac{1 + e^{-\gamma(t-t')} - e^{-\gamma t} - e^{-\gamma t'}}{\gamma} \right). \quad (5.6)$$

5.1. The equal time correlation function and the time-dependent diffusion coefficient

In the particular case $t = t'$, one gets from Eq. (5.1):

$$\langle x^2(t) \rangle = 2 \frac{kT}{m} t^2 E_{2-\delta,3} [-(\omega_\delta t)^{2-\delta}]. \quad (5.7)$$

The time-dependent diffusion coefficient, as defined by [9]

$$D(t) = \frac{1}{2} \frac{d}{dt} \langle x^2(t) \rangle, \quad (5.8)$$

is:

$$D(t) = \frac{kT}{m} t E_{2-\delta,2} [-(\omega_\delta t)^{2-\delta}]. \quad (5.9)$$

In the Ohmic case $\delta = 1$, Eqs. (5.7) and (5.9) reduce to the standard formulas

$$\langle x^2(t) \rangle = 2 \frac{kT}{m\gamma} t \quad (5.10)$$

and

$$D(t) = \frac{kT}{m\gamma} (1 - e^{-\gamma t}). \quad (5.11)$$

At large times (*i.e.* $\omega_\delta t \gg 1$), Eqs. (5.7) and (5.9) yield

$$\langle x^2(t) \rangle \simeq 2 \frac{kT}{m} \frac{1}{\omega_\delta^2} \frac{(\omega_\delta t)^\delta}{\Gamma(\delta + 1)} \quad (5.12)$$

and

$$D(t) \simeq \frac{kT}{m} \frac{1}{\omega_\delta} \frac{(\omega_\delta t)^{\delta-1}}{\Gamma(\delta)}. \quad (5.13)$$

Eqs. (5.12) and (5.13) show that, for $\omega_\delta t \gg 1$, the particle motion is subdiffusive for $0 < \delta < 1$ and superdiffusive for $1 < \delta < 2$. In view of this fact, the expression

(5.1) of $\langle x(t)x(t') \rangle$ shows that the displacement of a subdiffusive or of a superdiffusive particle is an aging variable, as it is the case for a normally diffusing particle [8]-[9].

5.2. The fluctuation-dissipation ratio

Concerning the particle displacement, which is not a variable at equilibrium, a modified FDT can be written as

$$\chi_{xx}(t, t') = \beta \Theta(t - t') X(t, t') \frac{\partial \langle x(t)x(t') \rangle}{\partial t'}, \quad (5.14)$$

where $\chi_{xx}(t, t')$ is the displacement response function [8]. We have previously demonstrated that, for a diffusing particle, the fluctuation-dissipation ratio $X(t, t')$ can be expressed in terms of the time-dependent diffusion coefficients $D(\tau)$ and $D(t_w)$, where $\tau = t - t'$ denotes the observation time and $t_w = t'$ the waiting time [9]-[10]:

$$X(\tau, t_w) = \frac{D(\tau)}{D(\tau) + D(t_w)}. \quad (5.15)$$

Using the above found expression (5.9) of the time-dependent diffusion coefficient, one gets:

$$X(\tau, t_w) = \frac{\tau E_{2-\delta, 2} [-(\omega_\delta \tau)^{2-\delta}]}{\tau E_{2-\delta, 2} [-(\omega_\delta \tau)^{2-\delta}] + t_w E_{2-\delta, 2} [-(\omega_\delta t_w)^{2-\delta}]}. \quad (5.16)$$

When $\delta = 1$, we recover the result we obtained in [9]:

$$X(\tau, t_w) = \frac{1 - e^{-\gamma \tau}}{2 - e^{-\gamma \tau} - e^{-\gamma t_w}}. \quad (5.17)$$

At large observation and waiting times (i.e. $\omega_\delta \tau \gg 1$, $\omega_\delta t_w \gg 1$), one can use in Eq. (5.16) the asymptotic expressions of $D(\tau)$ and $D(t_w)$ as given by Eq. (5.13). Eq. (5.16) then displays the fact that, in a subohmic or superohmic model of exponent δ , the large times aging regime is self-similar, as pictured by the fluctuation-dissipation ratio [10]:

$$X(\tau, t_w) \simeq \frac{1}{1 + \left(\frac{t_w}{\tau}\right)^{\delta-1}}. \quad (5.18)$$

Interestingly enough, $X(\tau, t_w)$ is then a function of t_w/τ , solely parametrized by δ . For $\delta = 1$, one retrieves the result $X = 1/2$ [8]-[9]. For any other value of δ , X is an algebraic function of t_w/τ . The limits $\tau \rightarrow \infty$ and $t_w \rightarrow \infty$ do not commute.

6. Conclusion

We have studied the two-time dynamics of a classical anomalously diffusing particle. The particle motion is governed by the generalized Langevin equation, with a noise of spectral density proportional to $\omega^{\delta-1}$ at low frequencies ($0 < \delta < 1$ or $1 < \delta < 2$). The Langevin force is modeled by a stationary Gaussian random process. The velocity and the displacement of the particle are themselves Gaussian processes, fully characterized by their averages and their two-time correlation functions.

Using a double Laplace transformation technique, we have been able to derive closed form analytic expressions for the correlation functions of the velocity and of the displacement. Both can be expressed in terms of Mittag-Leffler functions, which are thus demonstrated to play a central role, not only in the one-time dynamics as shown in [7], but also in the two-time dynamics of the anomalously diffusing particle. While the velocity thermalizes at large times (slowly, except for the standard Brownian motion case $\delta = 1$), the displacement never attains equilibrium and ages.

The thermal equilibrium velocity fluctuations decay in time following the same Mittag-Leffler law as the average velocity resulting from a given initial fluctuation. This result constitutes the generalization to the non-Ohmic case $\delta \neq 1$ of the regression theorem well-known for the exponential decay of fluctuations which holds in standard Brownian motion.

As for the particle displacement, we have obtained for the fluctuation-dissipation ratio characterizing aging an expression in terms of Mittag-Leffler functions, valid for any values of the observation and waiting times. For $0 < \delta < 1$ or $1 < \delta < 2$, the large times aging regime is described by a self-similar function of t_w/τ . This is in marked contrast to the standard Brownian motion case, in which the large times aging regime is described by the value $X = 1/2$ of the fluctuation-dissipation ratio.

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